# The Divergence Theorem

## Theorem (Stokes's Theorem)

*Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let* $\mathbf{F}$ *be a vector field whose components have continuous partial derivatives on an open region in* $\mathbb{R}^3$ *that contains $S$. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

# Divergence is infinitesimal flux

Given a vector field $\mathbf{F}$ and a point $(x_0, y_0, z_0)$, let $S_\varepsilon$ be the sphere of radius $\varepsilon$ centered at $(x_0, y_0, z_0)$. Define

$$I(\varepsilon) = \frac{1}{4\pi\varepsilon^2} \iint_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{S}.$$

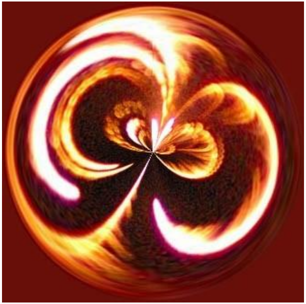
Then a straightforward (albeit *long*) calculation shows that

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \frac{1}{3} \operatorname{div} \mathbf{F}(x_0, y_0, z_0).$$

So div $\mathbf{F}$ is measuring the “net flux” around a point.

# Interpretation of the divergence

- div $\mathbf{F}(x_0, y_0, z_0) > 0$ means $(x_0, y_0, z_0)$ is a "source"
- div $\mathbf{F}(x_0, y_0, z_0) < 0$ means $(x_0, y_0, z_0)$ is a "sink"
- div $\mathbf{F} = 0$ means $\mathbf{F}$ is **incompressible**



# Statement of the Theorem

## Theorem (Gauß's Divergence Theorem)

*Let $E$ be a simple solid region and $S$ the boundary surface of $E$, given with the positive (outward) orientation. Let* $\mathbf{F}$ *be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

# Proof of the Theorem

See page 967

We must show

$$\iint_S (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \mathbf{n}\, dS = \iiint_E \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

It suffices to show

$$\iint_S P\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV \qquad \iint_S Q\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$$

$$\iint_S R\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

We'll show the last. The others are similar.

Assume $E$ is a region of type 1:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

Then $S = \partial E = S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{(x, y, z) \mid (x, y) \in D,\ z = \leq u_1(x, y)\}$$

$$S_2 = \{(x, y, z) \mid (x, y) \in D,\ z = \leq u_2(x, y)\}$$

$$S_3 = \{(x, y, z) \mid (x, y) \in \partial D,\ u_1(x, y) \leq x \leq u_2(x, y)\}$$

On the boundary surface $S_2$, which is the graph of $u_2$:

$$\mathbf{n}\, dS = \left( -\frac{\partial u_2}{\partial x}, -\frac{\partial u_2}{\partial y}, 1 \right) dA$$

So

$$\iint_{S_2} R\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA$$

On $S_1$, the outward normal points *down*, so

$$\iint_{S_1} R\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_{D} R(x, y, u_1(x, y))\,dA$$

Finally, on $S_3$, $\mathbf{n} \cdot \mathbf{k} = 0$, so

$$\iint_{S_3} R\mathbf{k} \cdot \mathbf{n}\, dS = 0$$

Putting it all together we have

$$
\begin{aligned}
\iint_S &= \iint_D \left(R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right) dA \\
&= \iint_D \int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}\, dz\, dA \\
&= \iiint_E \frac{\partial R}{\partial z}\, dV
\end{aligned}
$$

## Example (Worksheet Problem 1)

Use the Divergence Theorem to calculate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$ where

$$\mathbf{F} = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + yz^2\,\mathbf{k}$$

and $S$ is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$.

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## Solution

*We have* $\operatorname{div} \mathbf{F} = 2yz$ *so*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

$$= \int_0^1 \int_0^1 \int_0^2 2yz\, dz\, dy\, dx = 2.$$

## Example (Worksheet Problem 2)

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2 z)\,\mathbf{k}$$

and $S$ is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

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## Solution

*Let $E$ be the region bounded by $S$. We have*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E (x^2 + y^2)\, dV$$

*Switch to cylindrical coordinates and we get*

$$\iiint (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^{2} \int_{r^2}^{z} r^2 \cdot r\, dz\, dr\, d\theta = \frac{32\pi}{3}$$

## Theorem (Green's Theorem)

*Let $C$ be a positively oriented, piecewise smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then*

$$\int_C (P\,dx + Q\,dy) = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

*or*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D d\mathbf{F}\, dA$$

## Theorem (Stokes's Theorem)

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### Theorem (Gauß's Divergence Theorem)

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$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

# Manifold Destiny

- A **manifold** is a smooth subset of $\mathbb{R}^n$ (roughly speaking, something you can do calculus on)
- Examples: Solid regions, surfaces, curves
- A manifold can have a boundary, which is of one dimension lower.

- Each of the operators gradient, curl, divergence, and even the two-dimensional curl $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ are a kind of derivative.
- Each of the big three theorems takes the form

$$\int_{Q} d\Delta = \int_{\partial Q} \Delta,$$

where Q is a manifold, $\Delta$ is a vector field, $\int$ is whatever integral and $d$ is whatever derivative is appropriate from the context.

# But wait, there's more

- We can have zero-dimensional manifolds, too: they're points.
- The integral over a zero-dimensional manifold is just evaluation at the points.
- Other fundamental theorems in calculus can expressed using this extension of the $\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$ formalism

**Theorem (The Fundamental Theorem of Line Integrals)**

*Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$, and let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

**Theorem (The Fundamental Theorem of Calculus)**

*Let $F(x)$ be a function on $[a, b]$ with continuous derivative. Then*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$